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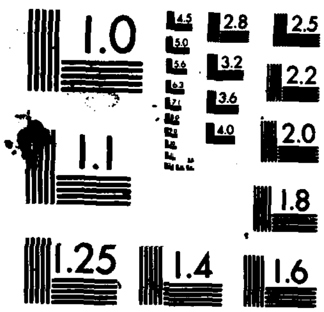

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Applied Mathematics

Brown University Providence RI

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PERTURBATION OF HOMOCLINICS AND SUBHARMONICS IN DUFFING'S EQUATION

by

Jack K. Hale and Adalberto Spezamiglio

March 1982 LCDS Report #82-3

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For (λ,μ) and (g,f) small, the equation

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$$\ddot{x} - x + 2x^3 + g(x) = -\lambda\dot{x} + \mu[\cos t + f(t)]$$

lambda, mu

is considered near the separatrix S of the unperturbed equation $(\lambda,\mu) = (0,0)$. For (g,f) in a neighborhood of the zero functions, a complete description (CONT)

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lambda
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is given of the bifurcation curves in (λ,μ) - space to homoclinic points. The perturbation of subharmonics outside S is also considered in a symmetric and nonsymmetric case.

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PERTURBATION OF HOMOCLINICS AND SUBHARMONICS IN DUFFING'S EQUATION

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March 1982

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PERTURBATION OF HOMOCLINICS AND SUBHARMONICS IN DUFFING'S EQUATION

by

Jack K. Hale and Adalberto Spezamiglio

ABSTRACT

For (λ,μ) and (g,f) small, the equation

$$\ddot{x} - x + 2x^3 + g(x) = -\lambda\dot{x} + \mu[\cos t + f(t)]$$

is considered near the separatrix S of the unperturbed equation $(\lambda,\mu) = (0,0)$. For (g,f) in a neighborhood of the zero functions, a complete description is given of the bifurcation curves in (λ,μ) - space to homoclinic points. The perturbation of subharmonics outside S is also considered in a symmetric and nonsymmetric case.

Perturbation of Homoclinics and Subharmonics in Duffing's Equation

I. Introduction

Recently, a considerable number of articles have been devoted to the study of strange attractors and how some of them arise near homoclinic orbits from successive subharmonic bifurcations. Different methods have been used.

In [2], the reduction of Liapunov-Schmidt is used in a second order equation with small time-periodic forcing and damping, such that the unperturbed equation has a homoclinic orbit Γ through zero. Regions in parameter space near zero are described where there are either homoclinic points or no homoclinic points for the perturbed equation near Γ. Those regions are defined by the bifurcation curves to homoclinic points, and the same analysis is done with respect to subharmonic solutions. Furthermore, it is proved that the bifurcation curves to subharmonics of order k approach the bifurcation curves to homoclinic points as $k \to \infty$. See also [1] for more details.

In [3], Greenspan and Holmes apply results based on Melnikov's method [6] in Duffing's equation perturbed by small dissipation and time-periodic forcing. In this case, the unperturbed equation has a pair of homoclinic orbits Γ^+, Γ^- through zero. Some calculations are shown with respect to homoclinic and subharmonic bifurcation curves. Also, perturbation of subharmonic solutions outside the separatrix $\Gamma^+ \cup \{0\} \cup \Gamma^-$ is considered.

The above papers may be used for further references. Here, we consider the perturbed Duffing equation

$$(1.1)\qquad \ddot{x} - x + 2x^3 + g(x) = -\lambda\dot{x} + \mu[\cos t + f(t)]$$

where $g(0) = 0$, g has continuous derivatives up through order two in $[-A,A]$, $A > 0$, f is continuous, 2π-periodic and $\rho = (\lambda,\mu)$ is a parameter in R^2. If $g = 0$, $f = 0$ and $\rho = 0$, then equation (1.1) has a pair of symmetric homoclinic orbits Γ^+, Γ^- through zero. We are concerned with the behavior of the solutions of (1.1) near the separatrix $S = \Gamma^+ \cup \{0\} \cup \Gamma^-$ for ρ, g and f small in a sense that will be specified. Our treatment follows [2].

We will denote by $C^k[a,b]$ the set of all functions q that are continuous together its derivatives up through order k in $[a,b]$, k a positive integer, with the C^k-norm $|q|_k$. $B(\mathbb{R})$ is the set of all continuous and bounded functions f on $\mathbb{R}$ with the supremum norm $|f|_\infty$, and P_T is the set of all continuous and T-periodic functions on $\mathbb{R}$, with the norm induced by $B(\mathbb{R})$. If f is in P_{2T}, we say that f is *odd harmonic in* P_{2T} if $f(t+T) = -f(t)$ for all t.

Consider now equation (1.1) with g an odd function and $|g|_2$ small. Then Γ^+, Γ^- are still symmetric, and if in addition f is odd harmonic in $P_{2\pi}$, that is, $f(t+\pi) = -f(t)$, then the bifurcation curves in ρ-space to homoclinic points in the left side of S near Γ^- coincide with the bifurcation curves to homoclinic points in the right side of S near Γ^+. That is, we have sectors in ρ-space defined by two curves C_m, C_M in which we have either homoclinic points in both sides near

S or no homoclinic points near S. In Theorem 1.1 we show that the set of pairs (g,f) near $(g,f) = (0,0)$ for which the above situation occurs is a manifold of codimension two in (g,f)-space. **Theorem 1.1 also gives a complete description of the curves of bifurcation to homoclinic orbits for (g,f) in a neighborhood of zero and not satisfying the above symmetry conditions.**

Under the above symmetric conditions on g and f, we consider in section 3 the perturbation of subharmonic solutions of odd order for the unperturbed equation outside S. (For subharmonics inside Γ^+ or Γ^-, see [2]). Here, we applied the results in [5]. We prove that the bifurcation curves C_m^k, C_M^k to subharmonics of order k, k odd, approach the bifurcation curves C_m, C_M, respectively, as $k \to \infty$.

Finally, the generic case when the bifurcation curves to homoclinic points in the left and right side do not coincide, is treated in section 4. We consider there the perturbation of subharmonics of even order outside S for the unperturbed equation. this case, we prove that the tangents to the bifurcation curves to subharmonics at $\mu = 0$ approach two curves that are not related to the bifurcation curves to homoclinic points as before. In fact, the limit lines will define a sector in parameter space near $\rho = 0$ in which we have subharmonic solutions outside S of even order k, for large k.

2. Perturbation of the homoclinics

Consider the second order equation

$$(2.1) \qquad \ddot{x} - x + 2x^3 + g(x) = -\lambda\dot{x} + \mu[\cos t + f(t)]$$

where g is in $C^2[-A,A]$ for a convenient $A > 0$, $g(0) = 0$, f is in $P_{2\pi}$ and $\rho = (\lambda,\mu)$ is a parameter in $\mathbb{R}^2$. When $g = 0$, $f = 0$ and $\rho = 0$, the point $(x,\dot{x}) = (0,0)$ is a saddle point, and the system associated to (2.1) has a pair of homoclinic orbits Γ^+, Γ^-, with α- and ω-limit sets being $\{0\}$. We are concerned with the behavior of the solutions of (2.1) in a neighborhood of $S = \Gamma^+ \cup \{0\} \cup \Gamma^-$, for $|\rho|$, $|g|_2$ and $|f|_\infty$ in a neighborhood of zero.

Let us describe the problem more specifically. If $|\rho|$, $|g|_2$ and $|f|_\infty$ are sufficiently small, equation (2.1) has a 2π-periodic solution $\phi = \phi(\rho,g,f)$ of small amplitude. Let $\gamma = \{(t,\phi(t),\dot{\phi}(t)) : t \in \mathbb{R}\}$ be its trajectory in $\mathbb{R}^3$ and let $S = S(\rho,g,f)$, $U = U(\rho,g,f)$ be respectively the stable and unstable manifolds in $\mathbb{R}^3$ of γ. We know that the sets

$$S(t) = \{(x,y) \in \mathbb{R}^2 : (t,x,y) \in S\},$$

$$U(t) = \{(x,y) \in \mathbb{R}^2 : (t,x,y) \in U\}$$

are periodic. We are interested in the existence of a transverse homoclinic point to $P_0 = (\phi(0),\dot{\phi}(0))$ in a neighborhood of S; that is, a point $P \neq P_0$, $P \in S(0) \cap U(0)$ where this intersection is transversal.

When $g = 0$ and $f = 0$, we can apply the results in [2] separately to each loop Γ^+ and Γ^-. For each one of them, we obtain a pair of differentiable curves through zero in

ρ-space, which define two sectors such that homoclinic points appear near one loop for ρ in one sector and no homoclinic point occurs for ρ in the other. As we shall see below, we can also give information about what happens in a neighborhood of zero in the (g,f)-space. In the statement of our result, figure 1 is helpful.

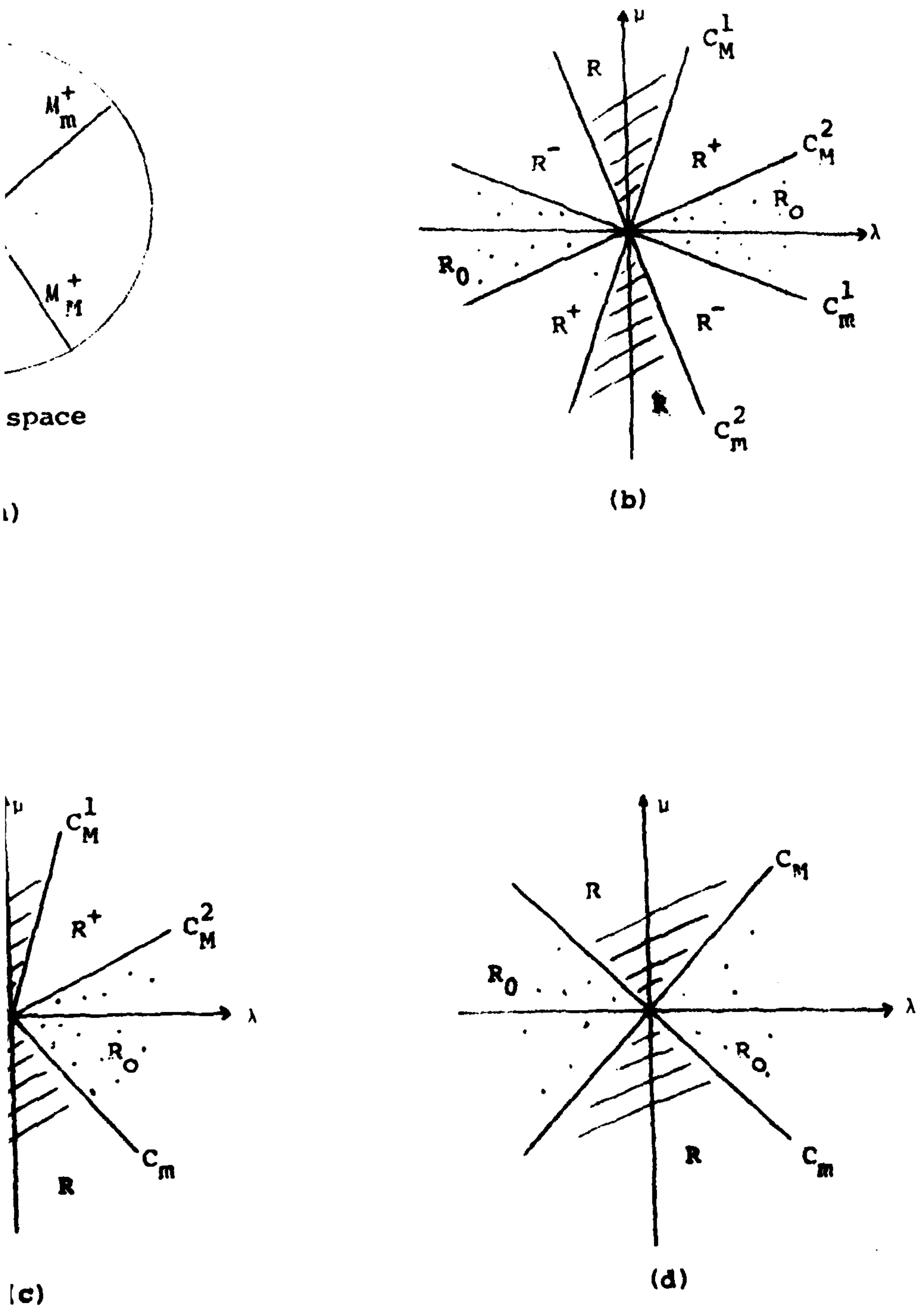
<u>Theorem</u> 2.1. There are neighborhoods U of S, V of $\rho = 0$, W of $(g,f) = 0$ and two submanifolds M_m, M_M in $C^2[-A,A] \times P_{2\pi}$ of codimension one, that divide W in four regions (fig. 1-a) and whose intersection M is a submanifold of codimension two that divides M_m, M_M in two components, such that:

(i) If $(g,f) \in W \setminus (M_m \cup M_M)$, there are two pairs of distint C^2-curves C_m^i, C_M^i, $i = 1,2$, which divide V in four sectors (fig. 1-b) such that equation (1) has homoclinic points in U in both sides of S for $\rho \in R$, only in the right side for $\rho \in R^+$, only in the left side for $\rho \in R^-$ and no homoclinic points in U for ρ in R_o.

(ii) If $(g,f) \in M_m^+$ then $C_m^1 = C_m^2 = C_m$ (fig. 1-c) and we have the same implications as in (i). Analogous statement holds for (g,f) in M_m^-.

(iii) If $(g,f) \in M_M$, the same analysis as in (ii) holds with respect the curves C_M^1 and C_M^2.

(iv) If $(g,f) \in M$ then we have the situation in fig. 1-d with the implications in (i).



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R
(c)
μ
C_M
R
R_0
λ
R_0
R
C_m
(d)

Figure 1

Proof: For $|g|_2$ small, the equation

$$\ddot{x} - x + 2x^3 + g(x) = 0$$)

a unique solution $p = p(g)$ satisfying $p(0) > 0$, $\dot{p}(0)=0, p(t) \to 0$ t $\to \pm \infty$. Let $\Gamma^+ = \{(p(t),\dot{p}(t)) : t \in \mathbb{R}\}$. If x is lution of (2.1), the change of varibles

$$x(t-\alpha) = p(t) + z(t)$$

2.1) leads to the following equation in z:

$$\ddot{z} + [-1 + 6p(t)^2 + g'(p(t))]z = F^+(t,z,\dot{z},\rho,\alpha,g,f)$$

2.3)

$$F^+(t,z,\dot{z},\rho,\alpha,g,f) \stackrel{\text{def}}{=} -\lambda\dot{p}(t) - \lambda\dot{z} + \mu[\cos(t-\alpha)+f(t-\alpha)]$$

$$- 6p(t)z^2 - 2z^3 - g(p(t)+z)$$

$$+ g(p(t)) + g'(p(t))z$$

Let $P = P_g$ be the continuous projection defined on) by

$$Ph = \dot{p}\int_{-\infty}^{\infty} \dot{p}(t)h(t)dt/\eta \quad , \quad \eta = \int_{-\infty}^{\infty} \dot{p}(t)^2 dt .$$

ma 2.1 in [2] now implies $S(0) \cap U(0)$ has elements near Γ^+ and only if

2.4)

(a) $PF^+(\cdot,z,\dot{z},\rho,\alpha,g,f) = 0$

(b) $z = K(I-P)F^+(\cdot,z,\dot{z},\rho,\alpha,g,f)$

re $Kh = K_g h$ is the unique bounded solution on $\mathbb{R}$ of the lation

$$\ddot{z} + [-1 + 6p(t)^2 + g'(p(t))]z = h(t),$$

in $\mathcal{B}(\mathbb{R})$, with initial value orthogonal to $(\dot{p}(0),\ddot{p}(0))$.

We can now apply the Implicit Function Theorem to tain $\delta > 0$, $\nu > 0$ such that equation (2.4-b) has a unique

solution $z_+^* = z_+^*(\rho,\alpha,g,f)$ for $|\rho| < \delta$, $|g|_2 + |f|_\infty < \delta$, $\alpha \in \mathbb{R}$, satisfying $z_+^*(0,\alpha,0,0) = 0$ and $|z_+^*| < \nu$. In fact, $z^*(0,\alpha,g,f) = 0$ for α in $\mathbb{R}$ and $|g|_2 + |f|_\infty < \delta$. Therefore, there exists a solution $x = x(\rho,\alpha,g,f)$ of (2.1) in the ν-neighborhood of Γ^+ if and only if

$$x(\cdot-\alpha) = p + z_+^*(\rho,\alpha,g,f)$$

where (ρ,α,g,f) is solution of the bifurcation equation

$$G^+(\rho,\alpha,g,f) \overset{\text{def}}{=} \frac{1}{\eta}\int_{-\infty}^{\infty} \dot{p}F^+(\cdot,z_+^*,\dot{z}_+^*,\rho,\alpha,g,f) = 0.$$

From the definition of F^+ in (2.3), the above equation has the form

$$G^+(\rho,\alpha,g,f) = -\lambda + \frac{\mu}{\eta}\int_{-\infty}^{\infty} \dot{p}[\cos(\cdot-\alpha) + f(\cdot-\alpha)] + G_0(\rho,\alpha,g,f) = 0$$

where $G_0(\rho,\alpha,g,f) = O(|\rho|^2)$ as $|\rho| \to 0$. Solving the above equation is equivalent to solve

$$H^+(\beta,\mu,\alpha,g,f) \overset{\text{def}}{=} -\beta + h^+(\alpha,g,f) + G_1(\beta,\mu,\alpha,g,f) = 0$$

where $H^+(\beta,\mu,\alpha,g,f) = G^+(\beta\mu,\mu,\alpha,g,f)/\mu$, $G_1(\beta,0,\alpha,g,f) = 0$ and

(2.5) $\quad h^+(\alpha,g,f) = \frac{1}{\eta}\int_{-\infty}^{\infty} \dot{p}[\cos(\cdot-\alpha) + f(\cdot-\alpha)]$.

An easy calculation shows that h^+ may be written in the form

$$h^+(\alpha,g,f) = \left(-\frac{1}{\eta}\int_{-\infty}^{\infty} p\cos\right)\sin\alpha + \frac{1}{\eta}\int_{-\infty}^{\infty} \dot{p}f(\cdot-\alpha)$$

and then $\partial h^+(\pi/2,0,0)/\partial\alpha = 0$, $\partial^2 h^+(\pi/2,0,0)/\partial\alpha^2 > 0$. Hence, if $\beta_m = h^+(\pi/2,0,0)$, we have $\partial H^+(\beta_m,0,\pi/2,0,0)/\partial\alpha = 0$, $\partial^2 H^+(\beta_m,0,\pi/2,0,0)\partial\alpha^2 > 0$. The Implicit Function Theorem implies there is a $\delta > 0$ and a unique function

$\alpha_m^+ = \alpha_m^+(\beta,\mu,g,f)$ for $|\beta - \beta_m|$, $|\mu|$ and $|g|_2 + |f|_\infty$ less than δ satisfying $\alpha_m^+(\beta_m,0,0,0) = \pi/2$, $\partial H^+(\beta,\mu,\alpha_m^+,g,f)/\partial\alpha = 0$. Thus, the function $M_m^+(\beta,\mu,g,f) = H^+(\beta,\mu,\alpha_m^+(\beta,\mu,g,f),g,f))$ is a minimum of H^+ with respect to α for the other variables fixed. Since $M_m^+(\beta_m,0,0,0) = 0$, $\partial M_m^+(\beta_m,0,0,0)/\partial\beta = -1$, the Implicit Function Theorem implies there is a unique function $\beta^*(\mu,g,f)$ for $|\mu|$, $|g|_2 + |f|_\infty$ less than δ satisfying $\beta^*(0,0,0) = \beta_m$, $M_m^+(\beta^*(\mu,g,f),\mu,g,f) = 0$. There are two solutions of $H^+(\beta,\mu,\alpha,g,f) = 0$ on one side of the curve $\beta = \beta^*(\mu,g,f)$ and no solutions on the other side. The function β^* defines the curve C_m^2 by the relation $\lambda = \beta^*(\mu,g,f)\mu$.

By exactly the same idea we obtain a function $\alpha_M^+(\beta,\mu,g,f)$ satisfying $\alpha_M^+(\beta_M,0,0,0) = 3\pi/2$, $\partial H^+(\beta,\mu,\alpha_M^+,g,f)/\partial\alpha = 0$ where $\beta_M = h^+(3\pi/2,0,0)$. For $\alpha = \alpha_M^+(\beta,\mu,g,f)$, **the function $H^+(\beta,\mu,\alpha,g,f)$ is a maximum. Following** the last paragraph, we obtain the curve C_M^2. In figure (1-b), those curves define the sector $R \cup R^+$ where we have homoclinic points near Γ^+ and $R_o \cup R^-$ where we do not have.

By using the same procedure with the solution $q(t)$ of equation (2.2) satisfying $q(t) \to 0$ as $t \to \pm\infty$, $q(0) < 0$, $\dot{q}(0)=0, \Gamma^- = \{(q(t),\dot{q}(t)) : t \in \mathbb{R}\}$, we obtain the bifurcation curves to homoclinic points in the left side of S, C_m^1 and C_M^1, respectively defined by the equations $M_m^-(\beta,\mu,g,f) = H^-(\beta,\mu,\alpha_m^-,g,f) = 0$ and $M_M^-(\beta,\mu,g,f) = H^-(\beta,\mu,\alpha_M^-,g,f) = 0$ where $\alpha_m^-(\gamma_m,0,0,0) = 3\pi/2$, $\alpha_M^-(\gamma_M,0,0,0) = \pi/2$, $\gamma_m = h^-(3\pi/2,0,0)$, $\gamma_M = h^-(\pi/2,0,0)$.

We will define now the manifolds M_m, M_M so that the analysis described in (i) becomes obvious. The curves C_m^1 and C_m^2 will coincide if and only if $F_m(\beta,\mu,g,f) =$ $= M_m^+(\beta,\mu,g,f) - M_m^-(\beta,\mu,g,f) = 0$ and $C_M^1 = C_M^2$ if and only if

$F_M(\beta,\mu,g,f) = M_M^+(\beta,\mu,g,f) - M_M^-(\beta,\mu,g,f) = 0$. Let us define

$$W = \{(g,f) : |g|_2 + |f|_\infty < \delta\},$$

$$\mathcal{M}_m = \{(g,f) \in W : F_m(\beta,\mu,g,f) = 0\},$$

$$\mathcal{M}_M = \{(g,f) \in W : F_M(\beta,\mu,g,f) = 0\}.$$

All the statements in Theorem 1.1 are now clear, except those concerning the codimensions. We will prove below that codim $\mathcal{M} = 2$ and we observe that the same proof shows that $\mathcal{M}_m$ and $\mathcal{M}_M$ have codimension one.

In order to show that codim $\mathcal{M} = 2$, let us consider two linearly independent directions from $(g,f) = 0$ defined as follows: we take

$$g(x) = g_a(x) = \tfrac{3}{2}ax^2,$$

$$f(t) = f_b(t) = b\sin 2t$$

for $|a|$ and $|b|$ sufficiently small. A straightforward calculation shows that, if

$$p_a(t) = 4[\sqrt{a^2+4}\,(e^t + e^{-t}) + 2a]^{-1},$$

then $p = p_a$ and $q = -p_{-a}$, and the functions F_m, F_M are given by

$$F_m(\beta,\mu,a,b) = -\tfrac{1}{\eta}[(\int_{-\infty}^{\infty} p_a\cos)\cos\theta_m^+ + 2b(\int_{-\infty}^{\infty} p_a\cos 2\cdot)\cos 2\theta_m^+]$$

$$+ \tfrac{1}{\zeta}[(\int_{-\infty}^{\infty} p_{-a}\cos)\cos\theta_m^- + 2b(\int_{-\infty}^{\infty} p_{-a}\cos 2\cdot)\cos 2\theta_m^-]$$

$$+ G_2^m(\beta,\mu,a,b),$$

$$F_M(\beta,\mu,a,b) = \frac{1}{\eta}[(\int_{-\infty}^{\infty} p_a \cos)\cos\theta_M^+ + 2b(\int_{-\infty}^{\infty} p_a \cos 2\cdot)\cos 2\theta_M^+]$$

$$- \frac{1}{\zeta}[(\int_{-\infty}^{\infty} p_{-a} \cos)\cos\theta_M^- - 2b(\int_{-\infty}^{\infty} p_{-a} \cos 2\cdot)\cos 2\theta_M^-]$$

$$+ G_2^M(\beta,\mu,a,b) ,$$

where $G_2^m(\beta,0,a,b) = G_2^M(\beta,0,a,b) = 0$. The expression $\theta_m^+ = \theta_m^+(\beta,\mu,a,b)$ is defined by $\alpha_m^+(\beta,\mu,a,b) = \pi/2 + \theta_m^+(\beta,\mu,a,b)$ and hence $\theta_m^+(\beta_m,0,0,0) = 0$. Analagous observations for the other θ's.

We must show that the Jacobian determinant $\det \partial(F_m,F_M)/\partial(a,b)$ is different from zero for $(a,b) = (0,0)$ and $|\mu|$ small. The partial derivatives evaluated at $(\beta,0,0,0)$ are given by

$$\frac{\partial F_m}{\partial a} = \int_{-\infty}^{\infty} \operatorname{sech}^2 \cos/\eta_o = - \frac{\partial F_M}{\partial a} ,$$

(2.6)

$$\frac{\partial F_M}{\partial a} = 4\int_{-\infty}^{\infty} \operatorname{sech}\cos 2\cdot/\eta_o = \frac{\partial F_M}{\partial b} .$$

where $\eta_o = \int_{-\infty}^{\infty} \dot{p}_o^2$. Clearly, we only have to show that both integrals in (2.6) are different from zero. Evaluating them by the method of residues, we obtain

$$\int_{-\infty}^{\infty} \operatorname{sech}\cos 2\cdot = 2\pi e^{-\pi}/1 + e^{-2\pi} ,$$

$$\int_{-\infty}^{\infty} \operatorname{sech}^2 \cos = \pi e^{-\pi/2}/1 - e^{-\pi} .$$

This completes the proof.

We observe that when $|g|_2$ and $|f|_\infty$ are sufficiently small, with $g(-x) = -g(x)$ and $f(t+\pi) = -f(t)$, then (g,f) is in M. In fact, one can prove in this case that the bifurcation functions satisfy $G^+(\lambda,\mu,\alpha+\pi,g,f) = G^-(\lambda,\mu,\alpha,g,f)$.

3. Perturbation of the subharmonics: the symmetric case

For the equation

$$(3.1) \qquad \ddot{x} - x + 2x^3 + g(x) = 0 ,$$

let p and q be as in the proof of Theorem 2.1, $p_\delta(t)$ the solution satisfying $p_\delta(0) = (1+\delta)p(0)$, $\dot{p}_\delta(0) = 0$ for $\delta > 0$ and $|g|_2$ in a neighborhood of zero. Then, p_δ is periodic with least period $\omega(\delta)$ and $\omega(\delta) \to \infty$ as $\delta \to 0$ and $(p_\delta(t), \dot{p}_\delta(t))$ lies outside S for all $t \in \mathbb{R}$.

If k is a positive integer, let δ_k be such that $\omega(\delta_k) = 2k\pi$ and $p_k = p_{\delta_k}$. For a 2π-periodic function F and a real number Λ, let

$$H^k(\alpha) = \int_{-k\pi}^{k\pi} \dot{p}_k(t) F(t-\alpha)\,dt ,$$

$$H^\Lambda(\alpha) = \int_{-\infty}^{\infty} [\dot{p}(t) + \Lambda \dot{q}(t)] F(t-\alpha)\,dt .$$

Lemma 3.1. If $F(t+2\pi) = F(t)$ is given, then for every $\varepsilon > 0$, there exists an integer K such that

$$\left| \frac{\partial^i H^k}{\partial \alpha^i}(\alpha) - \frac{\partial^i H^\Lambda}{\partial \alpha^i}(\alpha) \right| < \varepsilon \, |F|_\infty$$

for $i = 0,1,2$, if one the following conditions is satisfied:

(a) $k \geq K$, k odd, $\Lambda = -1$ and F odd harmonic in $P_{2\pi}$.

(b) $k \geq K$, k even and $\Lambda = 1$.

Proof: Let $\varepsilon > 0$ be given and suppose k odd, $\Lambda = -1$ and $F(t+\pi) = -F(t)$. We will prove first that

$$\left|\int_{-\infty}^{0} \dot{p}F(\cdot-\alpha) - \int_{0}^{\infty} \dot{q}F(\cdot-\alpha) - \int_{-k\pi}^{0} \dot{p}_k F(\cdot-\alpha)\right| < \frac{\varepsilon}{2}|F|_\infty$$

for $k \geq K$. The case $i = 0$ will be proved since the other part is symmetric.

Let $T = T(k)$ be the point in $(0,k\pi)$ for which $p_k(T) = 0$. By choosing K sufficiently large we have

$$\left|\int_{-\infty}^{-T} \dot{p}F(\cdot-\alpha) - \int_{T}^{\infty} \dot{q}F(\cdot-\alpha)\right| < \frac{\varepsilon}{4}|F|_\infty \quad , \quad k \geq K .$$

Now,

$$\left|\int_{-T}^{0} \dot{p}F(\cdot-\alpha) - \int_{0}^{T} \dot{q}F(\cdot-\alpha) - \int_{-k\pi}^{-T} \dot{p}_k F(\cdot-\alpha) - \int_{-T}^{0} \dot{p}_k F(\cdot-\alpha)\right| =$$

$$= \left|\int_{-T}^{0} \dot{p}F(\cdot-\alpha) - \int_{0}^{T} \dot{q}F(\cdot-\alpha) + \int_{0}^{-T+k\pi} \dot{p}_k(\cdot-k\pi)F(\cdot-\alpha) - \int_{-T}^{0} \dot{p}_k F(\cdot-\alpha)\right| .$$

By making the change of variables $x = p(t)$, $x = q(t)$, $x = p_k(t-k\pi)$ and $x = p_k(t)$ separately in each one of the above integrals and denoting the inverses respectively by $t^+(x)$, $t^-(x)$, $t_k^-(x)$ and $t_k^+(x)$, the last expression becomes

$$\leq \left|\int_{p(-T)}^{p(0)} F(t^+(\cdot)-\alpha) - \int_{0}^{p_k(0)} F(t_k^+(\cdot)-\alpha)\right| +$$

$$+ \left|-\int_{q(0)}^{q(T)} F(t^-(\cdot)-\alpha) + \int_{p_k(-k\pi)}^{0} F(t_k^-(\cdot)-\alpha)\right| .$$

If we fix σ, $0 < \sigma << 1$, then we have for the first modulus in the above expression, say I_1:

$$I_1 \le \left|\int_{p(-T)}^{\sigma} F(t^+(\cdot)-\alpha)\right| + \left|\int_{\sigma}^{p(0)} [F((t^+(\cdot)-\alpha) - F(t_k^+(\cdot)-\alpha)]\right| +$$

$$+ \left|\int_0^{\sigma} F(t_k^+(\cdot)-\alpha)\right| + \left|\int_{p(0)}^{p_k(0)} F(t^+(\cdot)-\alpha)\right| \le$$

$$\le \sigma|F|_\infty + \int_{\sigma}^{p(0)} |F(t^+(\cdot)-\alpha) - F(t_k^+(\cdot)-\alpha)| + \sigma|F|_\infty + \delta_k p(0)|F|_\infty .$$

Since $t_n^+(x) \to t^+(x)$ uniformly in compact sets and $\delta_n \to 0$ as $n \to \infty$, the last expression can be made less than $(\epsilon/8)|F|_\infty$ for large k and small σ. For the second modulus, similar estimates can be obtained taking into account that $t_n^-(x) \to t^-(x)$ uniformly in compact sets and $p_n(-n\pi) \to q(0)$ as $n \to \infty$.

For the case $i = 1$, we note that

$$H^k(\alpha) = \int_{-k\pi-\alpha}^{k\pi-\alpha} \dot{p}_k(\cdot+\alpha)F ,$$

and since $\dot{p}_k(k\pi) = \dot{p}_k(-k\pi) = 0$,

$$\frac{\partial H^k}{\partial\alpha}(\alpha) = \int_{-k\pi-\alpha}^{k\pi-\alpha} \ddot{p}_k(\cdot+\alpha)F .$$

Similarly,

$$\frac{\partial H^\Delta}{\partial\alpha}(\alpha) = \int_{-\infty}^{\infty} [\ddot{p}(\cdot+\alpha) - \ddot{q}(\cdot+\alpha)]F .$$

If $\tilde{F}$ is a smooth 2π-periodic approximation of F, we define

$$\tilde{H}^k(\alpha) = \int_{-k\pi-\alpha}^{k\pi-\alpha} \dot{p}_k(\cdot+\alpha)\tilde{F} ,$$

$$\tilde{H}^\Delta(\alpha) = \int_{-\infty}^{\infty} [\dot{p}(\cdot+\alpha) - \dot{q}(\cdot+\alpha)]\tilde{F} .$$

Hence,

$$\left|\frac{\partial H^k}{\partial\alpha}(\alpha) - \frac{\partial\tilde{H}^\Delta}{\partial\alpha}(\alpha)\right| \le |F - \tilde{F}|_\infty \int_{-k\pi}^{k\pi} |\ddot{p}| ,$$

$$\left|\frac{\partial H^\Delta}{\partial\alpha}(\alpha) - \frac{\partial\tilde{H}^\Delta}{\partial\alpha}(\alpha)\right| \le |F - \tilde{F}|_\infty \int_{-\infty}^{\infty} |\ddot{p} - \ddot{q}| .$$

If $\tilde{F}$ is sufficiently close to F, each of the above expressions can be made less than $(\varepsilon/3)|F|_\infty$. On the other hand,

$$\left|\frac{\partial H^\Delta}{\partial\alpha}(\alpha) - \frac{\partial\tilde{H}^k}{\partial\alpha}(\alpha)\right| =$$

$$= \left|\int_{-\infty}^{\infty} (\dot{p}-\dot{q})\frac{d}{d\alpha}\tilde{F}(\cdot-\alpha) - \int_{-k\pi}^{k\pi} \dot{p}_k \frac{d}{d\alpha}\tilde{F}(\cdot-\alpha)\right| .$$

The proof of part $i = 0$ shows that, for large k, the last expression can be made less than $(\varepsilon/3)|F|_\infty$.

Finally, for $i = 2$,

$$\frac{\partial^2 H^k}{\partial\alpha^2}(\alpha) = -\ddot{p}(k\pi)F(k\pi-\alpha) + \ddot{p}_k(-k\pi)F(-k\pi-\alpha) +$$

$$+ \int_{-k\pi-\alpha}^{k\pi-\alpha} \dddot{p}_k(\cdot+\alpha)F .$$

Since $\dddot{p}$ is continuous then $\partial^2 H^k/\partial\alpha^2$ is continuous and the same holds for $\partial^2 H^\Delta/\partial\alpha^2$. Similar estimates can be obtained taking into account that $\ddot{p}(-k\pi) = \ddot{p}_k(k\pi)$ is bounded as $k \to \infty$. Part (a) is then proved. The proof of part (b) is analogous, with obvious modifications. The proof is complete.

By using the same steps of the last proof, one can prove the following result:

<u>Lemma</u> 3.2. If p, q and p_k are as above and if $(F_k(t))$ is a uniformly bounded sequence with F_k in $P_{2k\pi}$ such that $F_k \to F$ uniformly in compact sets as $k \to \infty$, then for every $\varepsilon > 0$, there is a K such that

$$\left|\int_{-k\pi}^{k\pi} \dot{p}_k F_k - \int_{-\infty}^{\infty} [\dot{p} + \lambda\dot{q}]F\right| < \varepsilon$$

holds, if one of the conditions (a), (b) in Lemma 3.1 is satisfied. In (a), the condition on F is replaced by F_k *odd harmonic in* $P_{2k\pi}$.

Let us consider now the equation (2.1) with the symmetric conditions $g(-x) = -g(x)$ and $f(t+\pi) = -f(t)$. In this case, the solutions p and q of (2.2) defining Γ^+, Γ^- satisfy $q(t) = -p(t)$ for t in $\mathbb{R}$. We seek here subharmonics of order k for k odd, so let $h = h^+$ from Theorem 2.1, that is,

$$h(\alpha,g,f) = \frac{1}{\eta}\int_{-\infty}^{\infty} \dot{p}[\cos(\cdot-\alpha) + f(\cdot-\alpha)] .$$

We recall from that proof that $\partial h(\pi/2,0,0)/\partial\alpha = 0$, $\partial^2 h(\pi/2,0,0)/\partial\alpha^2 > 0$ and so there exists a unique function $\alpha_m^*(g,f)$ for $|g|_2 + |f|_\infty$ small, satisfying $\partial h(\alpha_m^*(g,f),g,f)/\partial\alpha = 0$, $\alpha_m^*(0,0) = \pi/2$. So, $\alpha_m^*(g,f)$ is a point of minimum for $h(\alpha,g,f)$. Also, there is a unique function $\alpha_M^*(g,f)$ with $\alpha_M^*(0,0) = 3\pi/2$ such that $\alpha_M^*(g,f)$ is a point of maximum for $h(\alpha,g,f)$.

For k an odd integer, let

$$(3.2) \qquad h_k(\alpha,g,f) = \frac{1}{\eta_k}\int_{-k\pi}^{k\pi} \dot{p}_k[\cos(\cdot-\alpha) + f(\cdot-\alpha)], \quad \eta_k = \int_{-k\pi}^{k\pi} \dot{p}_k^2 .$$

By Lemmas 3.1 and 3.2, there exists a K such that each h_k for $k \geq K$, k odd, has a minimum at $\alpha_m^k(g,f)$, a maximum at $\alpha_M^k(g,f)$ and $\alpha_m^k \to \alpha_m^*$, $\alpha_M^k \to \alpha_M^*$ as $k \to \infty$. Let us also assume here that

$$(3.3) \qquad \frac{d\omega}{d\delta}(\delta) < 0$$

for $\delta > 0$ in a neighborhood of zero. Let C_m and C_M be the bifurcation curves defined in Theorem 2.1, part (iv). We can now prove the following result:

Theorem 3.1. Under the above conditions, there are neighborhoods U of S, V of $\rho = 0$ and an integer K such that for any odd integer $k \geq K$, there are two C^2-curves C_m^k, C_M^k in V respectively tangents to $\lambda = h_k(\alpha_m^k)\mu$, $\lambda = h_k(\alpha_M^k)\mu$ at $\mu = 0$ so that those curves divide V into disjoint sectors R^k, R_o^k such that equation (2.1) has no subharmonics of least period $2k\pi$ in $U \times \mathbb{R}$ for $\rho \in R_o^k$ and at least two for $\rho \in R^k$. Furthermore, $C_m^k \to C_m$ and $C_M^k \to C_M$ as $k \to \infty$ along the odd integers.

Proof. In this proof, k will always be an odd integer. We choose K so that for $k \geq K$ the orbit $\Gamma_k = \{(p_k(t), \dot{p}_k(t)) : t \in \mathbb{R}\}$ is contained in the set U of Theorem 2.1. If $x(t)$ is a solution of (2.1), the change of variables

$$x(t-\alpha) = p_k(t) + z(t)$$

leads equation (2.1) to

$$\ddot{z} + [-1 + 6p_k(t)^2 + g'(p_k(t))]z = F_k(t,z,\dot{z},\rho,\alpha,\sigma,f)$$

(3.4)

$$F_k(t,z,\dot{z},\rho,\alpha,g,f) = -\lambda\dot{p}_k(t) - \lambda\dot{z} + \mu[\cos(t-\alpha) + f(t-\alpha)]$$

$$- 6p_k(t)z^2 - 2z^3 - g(p_k(t)+z) + g(p_k(t))$$

$$+ g'(p_k(t))z .$$

For the equation $\ddot{z} + [-1 + 6p_k(t)^2 + g'(p_k(t)]z = F(t)$ where F is $2k\pi$-periodic, hypothesis (3.3) implies it has a $2k\pi$-periodic solution if and only if

$$\int_{-k\pi}^{k\pi} \dot{p}_k(t)F(t)dt = 0 .$$

If P_k is the continuous projection defined on $P_{2k\pi}$ by

$$P_kF = \dot{p}_k\int_{-k\pi}^{k\pi}\dot{p}_kF/\eta_k \quad , \quad \eta_k = \int_{-k\pi}^{k\pi}\dot{p}_k^2 ,$$

then there is a unique $2k\pi$-periodic solution K_kF with initial value orthogonal to $(\dot{p}_k(0),\ddot{p}_k(0))$, and K_k defined on $(I-P_k)P_{2k\pi}$ is continuous and linear. Furthermore, there exist constants C and $\beta > 0$ independent of k such that

(3.5) $$|K_k(I-P_k)F|_1 \le Ce^{-\beta|t|}|F|_\infty \quad , \quad |t| \le k\pi$$

(see [1], Lemma 11.4.6). Therefore, equation (2.1) has a $2k\pi$-periodic solution near p_k if and only if

(a) $$P_kF_k(\cdot,z,\dot{z},\rho,\alpha,g,f) = 0$$

(3.6)

(b) $$z = K_k(I-P_k)F_k(\cdot,z,\dot{z},\rho,\alpha,g,f) .$$

We can now use the Contraction Mapping Principle to
ıation (3.6-b). By using estimates (3.5) and the fact
and $\dot{p}_k$ are uniformly bounded in $\mathbb{R}$, we obtain $\delta > 0$
0 independent of k such that equation (3.6-b) has a
olution $z_k^* = z_k^*(\rho,\alpha,g,f)$ for $|\rho| < \delta$, $|g|_2 + |f|_\infty < \delta$
ıg $|z_k^*| < \nu$. Thus, subharmonics of order k near Γ_k
rmined by the solutions ρ, α of the bifurcation

$$G_k(\rho,\alpha,g,f) = \frac{1}{\eta_k}\int_{-k\pi}^{k\pi} \dot{p}_k F_k(\cdot,z_k^*,\dot{z}_k^*,\rho,\alpha,g,f) = 0.$$

ore, the number ν can be chosen so that all
nics that are obtained in that way belong to the
hood U of Γ.

By the definition of F_k in (3.4), the above equation
form

$$-\lambda + \mu h_k(\alpha,g,f) + \tilde{G}_k(\rho,\alpha,g,f) = 0$$

$_k(\rho,\alpha,g,f) = O(|\rho|^2)$ as $|\rho| \to 0$. The bifurcation
C_m^k, C_M^k are now obtained by exactly the same
e as in Theorem 2.1. In order to prove the statement
rgence, we observe first that z_k^* is odd harmonic, and
$_k(\cdot,z_k^*,\dot{z}_k^*,\rho,\alpha,g,f)$. Furthermore, since $p_k \to p$ and
uniformly in compact sets as $k \to \infty$, then
$,\rho,\alpha,g,f) \to F^+(t,z,\dot{z},\rho,\alpha,g,f)$ uniformly for all the
s in compact sets, and from Lemmas 3.1 and 3.2,
F^+ uniformly in compact sets as $k \to \infty$, k odd. By
ty with respect to initial data, $z_k^* \to z_+^*$ and so
$\dot{z}_k^*,\rho,\alpha,g,f) \to F^+(t,z_+^*,\dot{z}_+^*,\rho,\alpha,g,f)$ uniformly in compact

sets. Lemmas 3.1 and 3.2 now imply $G_k(\rho,\alpha,g,f) \to G^{+}(\rho,\alpha,g,f)$ as $t \to \infty$, k odd, for $|\rho|$ and $|g|_2 + |f|_\infty$ sufficiently small, $|\alpha| \neq k\pi$. The proof is complete.

4. The nonsymmetric case

Let $p(g), q(g)$, $|g|_2$ sufficiently small, be the solutions of (2.2) defining the homoclinic orbits $\Gamma^+(g), \Gamma^-(g)$ in the proof of Theorem 2.1. Suppose g_0 is a fixed function, $\tilde{p} = p(g_0)$, $\tilde{q} = q(g_0)$, and g_0 satisfies the nonsymmetric condition

$$\int_{-\infty}^{\infty} [\tilde{p}(t) + \tilde{q}(t)] \cos t \, dt \neq 0 \,. \tag{4.1}$$

In order to fix ideas, let us suppose the integral in (4.1) is positive.

Our objective is to discuss the existence of subharmonic solutions of (2.1) of order k, k even, outside $\Gamma^+(g) \cup \{0\} \cup \Gamma^-(g)$ for g in a neighborhood of g_0 and f in a neighborhood of zero. To simplify notation, let $p = p(g)$, $q = q(g)$. From Lemma 3.1, we expect the curves of bifurcation of these subharmonic solutions to be related to the function

$$H(\alpha, g, f) = \frac{1}{\zeta}\int_{-\infty}^{\infty} (\dot{p} + \dot{q})[\cos(\cdot - \alpha) + f(\cdot - \alpha)] \,,$$

$$\zeta = \int_{-\infty}^{\infty} (\dot{p} + \dot{q})\dot{p} \,.$$

It is easy to see that H may be written in the form

$$H(\alpha, g, f) = -\left[\frac{1}{\zeta}\int_{-\infty}^{\infty} (p+q)\cos\right]\sin\alpha + \frac{1}{\zeta}\int_{-\infty}^{\infty} (\dot{p}+\dot{q}) f(\cdot - \alpha) \,.$$

We then notice that $\partial H(\pi/2, g_0, 0)/\partial\alpha = 0$, $\partial^2 H(\pi/2, g_0, 0)/\partial\alpha^2 > 0$ and so there is a unique function $\alpha_m^*(g,f)$ defined for $|g - g_0|_2 + |f|_\infty$ small satisfying $\alpha_m^*(g_0, 0) = \pi/2$ and $\alpha_m^*(g,f)$ is a point of minimum for $H(\alpha, g, f)$. Also, there is a unique function $\alpha_M^*(g,f)$ for $|g - g_0|_2 + |f|_\infty$ small satisfying $\alpha_M^*(g_0, 0) = 3\pi/2$ and $\alpha_M^*(g,f)$ is a point of maximum for $H(\alpha, g, f)$.

For k an even integer, let $H_k(\alpha,\sigma,f)$ be defined by formula (3.2). By Lemmas 3.1 and 3.2, there exists an integer K such that each function H_k, for $k > K$, k even, has a minimum at $\alpha_m^k(\sigma,f)$, a maximum at $\alpha_M^k(\sigma,f)$ and $\alpha_m^k \to \alpha_m^*$, $\alpha_M^k \to \alpha_M^*$ as $k \to \infty$. We will also assume that hypothesis (3.3) is satisfied. By exactly the same procedure as in Theorem 3.1, we can prove.

Theorem 4.1. Under above conditions, there are neighborhoods U of S, V of $\rho = 0$ and an integer K such that for any even integer $k > K$, there are two C^2-curves C_m^k, C_M^k in V respectively tangents to $\lambda = H_k(\alpha_m^k)\mu$, $\lambda = H_k(\alpha_M^k)\mu$ at $\mu = 0$ so that those curves divide V into disjoint sectors R^k, R_o^k such that equation (2.1) has no subharmonics of least period $2k\pi$ in $U \times \mathbb{R}$ for ρ in R_o^k and at least two for ρ in R^k. Furthermore, the above tangents approach $\lambda = H(\alpha_m^*)\mu$, $\lambda = H(\alpha_M^*)\mu$ respectively as $k \to \infty$.

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